Using Geometric Probability: Until We Meet — Grade Ten

Ohio Standards Connections

Data Analysis and Probability

Benchmark J
Compute probabilities of compound events, independent events, and simple dependent events.
(Grades 8 - 10)

Indicator 7
Model problems dealing with uncertainty with area models (geometric probability).
(Grade 10)

Mathematical Processes Standard

Benchmark B
Apply mathematical knowledge and skills routinely in other content areas and practical situations.
(Grades 8 - 10)

Lesson Summary:
Students will conduct a simulation of a real-world situation and construct a mathematical model using geometric probability to find the probability of the occurrence of an event. During the lesson, students will encounter actual and theoretical probabilities and have the opportunity to consider the usefulness of each within the context of the situation.

This lesson is most appropriate after an introduction to geometric probability.

Estimated Duration: *100 minutes*

Pre-Assessment:
Students should be able to plot ordered pairs, find areas of simple geometric shapes, graph linear inequalities, and find basic geometric probabilities.

- A sample assessment has been included via Blackline Master #1 (Pre-Assessment: Basic Geometric Probability). Use this assessment to evaluate student understanding of the basics of geometric probability using area calculations of simple shapes.
- Sample tasks that can be used to assess plotting points and making graphs can be easily found in a variety of instructional materials.

Scoring Guidelines:
Scoring should be informal. Walk around the room and observe the answers students record on the Pre-Assessment Blackline Master #1. (The correct answers are $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{4}$.)

Instructional Tip:
Grade 10 students will be able to determine the probabilities for the three problems on Blackline Master #1 if they have been exposed to geometric probability. However, if students are having difficulty, the teacher may consider one the following interventions or an alternate intervention strategy:

- *Interactively demonstrate how to calculate geometric probability for the circle problem (area of shaded circle ÷ area of whole circle) by encouraging student participation through questioning. Then, encourage students to go back and try the other two questions again.*
- *Facilitate interactive instruction through an appropriate question and answer discussion with students. Sample questions might include:*
  - *Could someone describe how he/she determined the probability for problem number ___?*
  - *What do you suppose is meant by geometric probability?*
  - *Who can describe something about simple probability? (Take student's description and relate it to geometric probability.)*
  - *What is the area of the shaded region? The clear region?*
  - *What should be done with the area calculations?*

Post-Assessment:

Other Related Ohio Standards

Measurement

Benchmark A
Solve increasingly complex non-routine measurement problems and check for reasonableness of results. (Grades 8 - 10)

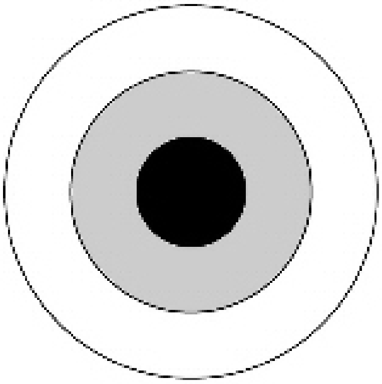
Patterns, Functions and Algebra

Benchmark D
Use algebraic representations, such as tables, graphs, expressions, functions and inequalities, to model and solve problem situations. (Grades 8 - 10)

- Students will create a geometric probability game. A sample game is included in the lesson.
- The game students create should not be too similar to any of the games used during the lesson.
- Students will set the scoring rules for the game, and they will calculate the probability of winning.
- Students should be encouraged to include dependent and independent events, such as the probability of winning two times in a row or of player A winning a game, then of player B winning a game.
- Students may also include probabilities of compound events, such as the probability of a game piece landing in the grey area or the white area.

Teacher Tip:
Below is a sample game and the scoring criteria for the game. Use this example to help you determine appropriate directions for students. This example has been provided to clarify the level of expectation.

A sample game board might look as follows:



The radius of the outer circle is 10cm, the radius of the middle circle is 7cm, and the radius of the inner circle is 4cm.

The scoring criteria are as follows:
- *Each player chooses a primary and a secondary color.*
- *Players take turns tossing a penny onto the game board.*
- *Score 2 points for hitting the primary color and 1 point for hitting the secondary color.*
- *The first player to reach 10 wins.*

Note: You and the students would need to determine what happens when a penny lands on multiple colors based upon the scope of experience students have had dealing with probability.

Scoring Guidelines:
Sample Rubric for Post-Assessment:

4 = Student creates a game environment (i.e., game board or game scenario and scoring criteria) written clearly and readily understood.

- Student calculates area correctly for all pertinent geometric shapes.
- Student uses the areas to calculate geometric probabilities accurately.
- Changing the geometric shapes or sizes will impact the way the game is scored.

3 = Student creates a game environment (i.e., game board or game scenario and scoring criteria) written clearly and readily understood.

- Student calculates areas correctly for all pertinent geometric shapes.
- Student uses the areas to calculate geometric probabilities accurately.
- Student work lacks evidence of understanding how to interpret probabilities (i.e., changing the geometric shapes or sizes will not impact how the game is scored).

2 = Student creates a game environment (i.e., game board or game scenario and scoring criteria) written clearly and readily understood.

- Student calculates areas correctly for most of the pertinent geometric shapes.
- Student uses the areas to calculate geometric probabilities with errors.
- Student work lacks evidence of understanding how to interpret probabilities.

1= Student creates a game environment (i.e., game board or game scenario and scoring criteria), but is not clearly written or is not readily understood.

- Student calculates areas for most of the pertinent geometric shapes with errors.
- Student uses the areas to calculate geometric probabilities with errors.
- Student work lacks evidence of understanding of how to interpret probabilities.

0 = No evidence is available of student's understanding of this assignment.

Teacher Tip:
Adjust the rubric as needed to include scoring criteria for other specific objectives that you include based on your students' understanding and history with probability (dependent, independent and/or compound probabilities).

Instructional Tip:
This assessment is ideal for a project that students work on over time with scheduled checkpoints using the sample rubric as a guide.

Score	Ideas for student – teacher interaction and action
3	Student receives constructive feedback from teacher for improving weaknesses.
2	Student receives intervention for calculations, feedback on existing work, and assistance as appropriate and resubmits work prior to final scoring.
1	Student conferences with teacher to consult about game ideas, receives advice for moving the project forward, and schedules at least two more conferences to checkpoint progress.
0	Student conferences with teacher to determine appropriate

	action.

Instructional Procedures:

1. Introduce the lesson with a problem situation: Mr. Xander and Ms. Yonder have to share a computer for inputting attendance data. Both of them are free for the period from 2:30 until 3:10 pm. Each one takes ten minutes to do the daily attendance. To meet their time restrictions on the computer, the teachers must arrive between 2:30 and 3:00 pm. What is the probability one of them will have to wait for the other to use the terminal?
2. Facilitate a discussion about ways to approach the problem. If the idea of finding times when the pair are both in the room does not occur, ask probing questions to focus student thinking.
3. Introduce the idea of creating a simulation of teacher arrivals for inputting attendance data. Suggest that it would simplify the problem to develop a way to indicate the time Mr. Xander and Ms. Yonder arrive at the computer. This can be done by referencing arrival times based upon the number of minutes after 2:30 pm for each teacher. So, students will generate pairs of random numbers, each from 0 to 30, to simulate Mr. Xander and Ms. Yonder arriving at the computer. Give an example for students to ensure that they understand how to interpret the data.

Teacher Tip:
The simulation can be modeled using random number pairs as follows: interpret the number pair (0, 15) as Mr. Xander arrives at 2:30 (2:30 + 0 min.) and Ms. Yonder arrives at 2:45 (2:30 + 15 min.), there is no conflict. The point (5, 9) would be interpreted as Mr. Xander arrives at 2:35 and Ms. Yonder arrives at 2:39, creating a conflict.

Random number generators are available using technology such as graphing calculators or computer programs. (Many graphing calculators are equipped or may be loaded with programs to generate random numbers.) Or you could use more traditional methods like rolling number cubes (use six number cubes and subtract the sum of the cubes from 6 to generate numbers between 0 and 30).

It may be useful to provide a table or have students design a table prior to beginning the data collection process.

4. Organize students into pairs to begin the data collection. Student pairs will generate and record 25 pairs of arrival times using the specified simulation method. Students will use their data to calculate the experimental probability for the occurrence of conflicts between Mr. Xander and Ms. Yonder having access to the computer.
   Experimental probability = (# of conflicts) ÷ (total # of data pairs).

5. Ask students to consider a way to find the theoretical probability of there being a conflict between Mr. Xander and Ms. Yonder's computer times. Students may generate some interesting ideas to try. If no one suggests using geometric probability, use questioning strategies or offer the idea and solicit student's feedback.

<u>**Commentary:**</u>
Students have collected the data, but they need to make a transition from what they already know to learning an approach for finding the theoretical probability of Xander and Yonder having a time overlap. Now may be a good time to invite students to take some time to consider geometric probability as an option and to explore the accuracy of experimental vs. theoretical probability. These connections may be made through playing a game.

A mini-lesson follows to help students make the needed connections. You may decide that this mini-lesson is not needed. Another alternative is to change when this mini-lesson is taught. The author of this lesson thought it appropriate to break up the problem at this point; however, you should make the appropriate decision for your students based on their level of understanding.

<u>Mini-lesson:</u> Theoretical vs. Experimental Geometric Probability

- Provide students with several shapes with regions shaded (something similar to a large version of one of the problems used for the pre-assessment).
- Students calculate theoretical probability of landing on any part of the shaded area by dividing the area of the shaded region by the area of the whole shape.
- Organize students in pairs to toss pennies to find the experimental probability of landing in shaded regions and unshaded regions. Allow students to play for about 5 – 10 minutes.
- Invite students to explore the accuracy of their experimental probabilities by comparing: 1) the experimental probability calculated using only data generated by their team vs. the theoretical probability; 2) the experimental probability calculated using data generated by five teams vs. the theoretical probability; and 3) the experimental probability calculated using the whole class' data vs. the theoretical probability.
- Ask students, after they have considered the three choices, to determine which experimental probability was closest to the theoretical probability and why.

Now students are ready to proceed with the Xander/Yonder problem.

6. Ask students how to proceed with the Mr. Xander and Ms. Yonder problem if they want to have the most accurate experimental probability. The class should conclude that they need to combine their data in order to increase the accuracy of the experiment.

7. Facilitate the combining of the class' data. This should include the number of times of conflict and the number of times of no conflict. Then, instruct the students to calculate the experimental probability for the occurrence of time conflicts between Mr. Xander and Ms. Yonder. Facilitate sharing to reach an acceptable result for the class' data.

8. Ask students if geometric probability could be used if the situation could be adequately modeled. The class should conclude that geometric probability is an option.

9. Organize the students into groups of four. Distribute graph paper to each group.

10. Instruct the groups to prepare the graph paper as follows:
    - Draw a boundary around a 30 unit by 30 unit square grid. (Note: each interval represents one minute.)
    - Ask students how many possible ordered pairs are on the graph if only integers are considered. There are 900 possible ordered pairs using only integer values, and that represents the area of the total region for the graph (30 x 30).

Instructional Tip:

Students may need help connecting the total number of ordered pairs to the area of the region or that the graph's boundary will serve as a region for the calculation of the geometric probability. Circulate the room and use questioning to help student groups determine this on their own.

Students will need to make the connection of considering the simple line graph they are creating in a new way, outside the bounds of its traditional interpretation of finding and analyzing slopes, y-intercepts, etc. They will need to see the range and domain as boundaries for a region and consider the graph itself to be defining smaller regions within the larger region.

The final goal will be for the students to realize that they will need to find the area of specific regions. However, they will first have to use traditional methods to interpret the meaning of the graph so they can identify which regions will be pertinent to their geometric probability calculations.

Circulate strategically to enable you to visit those groups that require more help sooner and those that are more self sufficient later. Try to visit all groups before it is time for the students to calculate the geometric probability using the region they graphed.

- Label the x-axis time Mr. Xander arrives and the y-axis time Ms. Yonder arrives.
- Students graph only those ordered pairs that occur when there is a conflict. The four students should have at least 50 numbers.
- Students should examine the 30 X 30 grid and look for a pattern formed by the points that have been plotted. The students should mark

additional points as required to complete the pattern that was started.
- Students should count the points that make up the completed pattern that represents points of conflict between Mr. Xander and Ms. Yonder.
- An estimate of the probability is made when the number of points marked is used as a numerator and 900 is used as the denominator.

11. Ask students if Mr. Xander arriving at 2:30.45 pm and Ms. Yonder arriving at 2:40.56 pm is included in their model. A brief discussion of time being continuous as opposed to discrete can happen here, or just discuss the concept that time is continuous. This is the final step of the model.

12. Ask students to adjust their models so they include all possible times. A good suggestion is to tell students to let y stand for the time Ms. Yonder arrives. Ask them to discuss in their groups and to determine an appropriate expression for the times when Mr. Xander (x) would be in the room that would cause a conflict. After waiting sufficiently for students to discuss, guide them, as needed, to the following conclusions: Mr. Xander can be in the room 10 minutes before time y; i.e., $x – 10$, and 10 minutes after time y; i.e., $x + 10$. So, there are a pair of inequalities, $y > x - 10$ and $y < x + 10$, or $x - 10 < y < x + 10$.

13. Distribute another piece of graph paper or instruct students to draw a separate grid on their first piece of graph paper to find the theoretical probability as follows:
    - Start with a coordinate plane that will accommodate x-values to 30 and y-values to 30.
    - Draw a boundary around the 30 by 30 grid square.
    - Shade in the region that solves the inequality, $x - 10 < y < x + 10$.
    - The students must find the area of the shaded region. Students can partition the shaded region into triangles and a parallelogram or can find the areas of the unshaded triangles, which are subtracted from 900 (the area of the square).
    - The theoretical probability is found by placing the area of the shaded region as the numerator and 900 as the denominator.

14. Compare this probability result with the previous methods: 1) the experimental probabilities calculated with a partner using 25 random values, 2) the experimental probability calculated using the data from the entire class, and 3) the geometrical probability approximation using discrete time and completing the pattern.

15. Organize students into groups of three to six students and ask each group to compare and contrast the methods used, that is, consider which method yielded the most accurate results and when would it be appropriate to use the different methods.

16. Facilitate a whole class discussion to summarize the lesson, to enable sharing of ideas, and to move the class to some reasonable conclusions.

17. Introduce a more complex version of the problem if students require additional work. Mr. Xander and Ms. Yonder are asked to keep track of parent contact in the data base in addition to the attendance. Now, each one takes fifteen minutes to record the appropriate data. Mr. Xander says, "Now

we will always have a conflict about using the computer." Is he correct?

18. Organize students in pairs to work on this problem. They should construct a geometric model that shows this problem and calculate the probability of a conflict. Then they should interpret their findings to determine if Mr. Xander is correct.

19. Review homework each day and provide time for students to play and critique classmates' games. The suggestions may be used to improve their games before collecting them for grading.

Differentiated Instructional Support:

Instruction is differentiated according to learner needs, to help all learners either meet the intent of the specified indicator(s) or, if the indicator is already met, to advance beyond the specified indicator(s).

- For students who struggle with finding areas of basic shapes and/or simple probability, partner them with others that possess understanding. Use problems from a mathematics text book or work book. Encourage study partners to work on problems outside of instructional time (e.g., prior to or after the daily lesson, during shared study hall periods, before or after school, etc.).
- For students experiencing difficulty with the algebra encountered in the geometric probability modeling, start with a simpler situation, such as: A spinner with four numbers: 0, 1, 2 and 3. If you spin it twice, what is the probability that the sum of the two numbers is less than 5? This will have a similar set up, but fewer options. Students can plot the ordered pairs similar to the method used in the instructional section. The pairs will be defined by: (1^{st} spin, 2^{nd} spin) and only those ordered pairs whose sum was less than 5 will be plotted within the 4 X 4 grid. The resulting theoretical probability will be 9/16. Similar probability problems yielding fractional values could be posed using linear graphs. For this particular situation the graph is the triangle formed by the inequality $y < 5 - x$ and the x and y axes.
- Some teachers may decide to expand this lesson to incorporate the above situation first, then follow this basic application with the more complex situation faced by Mr. Xander and Ms. Yonder.
- Students can investigate games of chance. Students can calculate probabilities of hitting certain regions of a bull's-eye on a dartboard. Students can assign winning payoffs on a game and discuss fair games and expected outcome.

Extension:

These are ideas for all students to continue learning on this topic - in the classroom or outside of the classroom.

- Students can solve similar but more difficult geometric probability problems such as: Joanne is going to rent a car at the airport. The courtesy van for car rental company A arrives every 10 minutes, while the courtesy van for car rental company B arrives every 15 minutes. What is the probability that Joanne waits less than five minutes to see both vans?

Solution: On the x-axis plot the waiting time for car rental company A and on the y-axis plot the waiting time for car rental company B. These should be a vertical line at 10 and a horizontal line at 15. Then, plot the lines for Joanne waiting five minutes or less for car rental company A (vertical line at $x = 5$ shaded to the left) and car rental company B (horizontal line at $y = 5$ shaded down). The area of intersection has an area of 25 units while the entire rectangle has an area of 150 units, so the theoretical geometric probability is $\frac{25}{150}$ or $\frac{1}{6}$.

What is the probability Joanne waits less than five minutes to see one of the vans?

Solution: Use a 10 by 15 grid. Joanne will not need more than 10 minutes to see one of the vans (that is van A), and the second van is on 15 minute cycles. Make a line at each of the five minute marks as before. Shade each as before. These are the probabilities of waiting five or less minutes for either van. Each one is a 5 by 15 rectangle (75 square units), while the other is a 5 by 10 rectangle (50 square units). The total area covered is 125 square units of the 150 square units in the "wait time" rectangle. However, the period of five minutes or less has been counted twice, once in each rectangle, so the area of 25 square units needs to be subtracted once to account for the doubling. That means 100 square units of area are used. So, 100 of 150 area units are covered for a probability of $\frac{100}{150}$ or $\frac{2}{3}$

Homework Options and Home Connections:

- Students create their own game and determine probabilities for winning using geometric probability. Students should design a game board or a game scenario that can be modeled using geometric probability.

Teacher Tip:
Set aside class time for students to play their games and to determine if the experimental probability matches the theoretical probability that they calculated.

- Students determine probabilities associated with winning the game multiple times in a row or having specific players win in a predetermined order.

Instructional Tip:
It is recommended that this task not be assigned until after the students have had the opportunity to explore experimental probability vs. theoretical probability. This task may be assigned multiple times during the course of this lesson, especially if the lesson spans multiple days. For each assignment the requirements should be altered to force practice of new understandings from the day's class work. Each time this task is assigned the teacher should create specific problems. In the beginning, be

more specific and split the work for the student, and over time provide less specific directions and guidance. Less specificity may encourage students to develop more creative strategies and solutions.

Key Vocabulary:
- geometric probability
- experimental probability
- theoretical probability
- independent
- dependent
- compound
- discrete

Technology Connections:
- Some graphing calculators can graph linear inequalities with the shading.
- Use a calculator that has a random generation feature.

General Tips:
Teachers should familiarize themselves with the technology prior to using it for this lesson.

Research Connections:
Geometric Probability, Faculty of the North Carolina School of Science and Mathematics, NCTM, 1988

Materials/Resources Needed:

For the teacher: Blackline Master #1(Pre-Assessment: Basic Geometric Probability), graphing calculator, other random number generation device, or a list of random numbers (presented so that all students can see). Note: The graphing calculator may also be used as a demonstration tool for the simulation used in this lesson.

For the students: Graph paper, ruler, random number table or technology with random number generator.

Attachments:
- Blackline Master #1: Pre-Assessment